1. Evaluate, $\int_0^{\pi} \frac{dx}{a + b\cos x}$ $a > 0,\ |b| < a$. Hence prove that :

(i) $\int_0^{\pi} \frac{\cos x}{(a + b\cos x)^2} \cdot dx = -\frac{\pi b}{(a^2 - b^2)^{3/2}}$.

(ii) $\int_0^{2\pi} \frac{dx}{(5 + 4\cos x)^2} = \frac{10\pi}{27}$.

(May 95, Dec. 98,

:

Let $I = \int_0^{\pi} \frac{dx}{a + b\cos x}$

Let $t = \tan\frac{x}{2}$

x	0	π
t	0	∞

$$\therefore \quad dx = \frac{2\,dt}{1+t^2};$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\therefore \quad I = \int_0^\infty \frac{1}{a+b\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2\,dt}{(1+t^2)}$$

$$= 2\int_0^\infty \frac{dt}{(a+b)+(a-b)t^2} = \frac{2}{a-b}\int_0^\infty \frac{dt}{t^2+\left(\frac{a+b}{a-b}\right)}$$

$$= \frac{2}{a-b}\left[\sqrt{\frac{a-b}{a+b}}\tan^{-1}\left(t\sqrt{\frac{a-b}{a+b}}\right)\right]_0^\infty$$

$$= \frac{2}{\sqrt{a^2-b^2}}\left[\frac{\pi}{2}\right] = \frac{\pi}{\sqrt{a^2-b^2}}$$

$$\therefore \quad \int_0^\pi \frac{dx}{a+b\cos x} = \frac{\pi}{\sqrt{a^2-b^2}} \qquad \ldots(1)$$

We choose b as a parameter and differentiate both sides w.r.t. b,

$$\int_0^\pi \frac{\partial}{\partial b}\left(\frac{1}{a+b\cos x}\right)\cdot dx = \pi\,\frac{d}{db}\left[(a^2-b^2)^{-1/2}\right]$$

$$\text{i.e.} \int_0^\pi \frac{-\cos x}{(a+b\cos x)^2}\,dx = \frac{\pi b}{(a^2-b^2)^{3/2}}$$

$$\text{i.e.,} \quad \int_0^\pi \frac{\cos x}{(a+b\cos x)^2}\,dx = \frac{-\pi b}{(a^2-b^2)^{3/2}} \qquad \ldots(2)$$

(ii) Again, differentiate (1) w.r.t. a, we get,

$$\int_0^\pi \frac{\partial}{\partial a}\frac{1}{(a+b\cos x)}\cdot dx = \frac{d}{da}\left[\frac{\pi}{\sqrt{a^2-b^2}}\right]$$

$$\therefore \quad -\int_0^\pi \frac{1}{(a+b\cos x)^2}\cdot dx = -\frac{\pi}{2}(a^2-b^2)^{-3/2}\cdot(2a)$$

$$\therefore \int_0^{\pi} \frac{dx}{(a+b\cos x)^2} = \frac{\pi a}{(a^2-b^2)^{3/2}} \qquad ...(3)$$

Now, $$\int_0^{2\pi} \frac{dx}{(a+b\cos x)^2} = \int_0^{\pi} \frac{dx}{(a+b\cos x)^2} + \int_0^{\pi} \frac{dx}{[a+b\cos(2\pi - x)]^2}$$

$$\left(\text{Using} \int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx\right)$$

$$= \int_0^{\pi} \frac{dx}{(a+b\cos x)^2} + \int_0^{\pi} \frac{dx}{(a+b\cos x)^2} = 2\int_0^{\pi} \frac{dx}{(a+b\cos x)^2}$$

From (2), $$\int_0^{2\pi} \frac{dx}{(a+b\cos x)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$$

putting $a = 5$, $b = 4$;

$$\int_0^{2\pi} \frac{dx}{(5+4\cos x)^2} = \frac{2\pi(5)}{(25-16)^{3/2}} = \frac{10\pi}{27}$$

Q 2) $$\int_0^{\pi} \frac{dx}{\alpha - \cos x} = \frac{\pi}{\sqrt{\alpha^2 - 1}}$$

Differentiating both sides w.r.t. α; we get,

$$\int_0^{\pi} \frac{-1}{(\alpha - \cos x)^2}\,dx = \frac{d}{d\alpha}\left(\frac{\pi}{\sqrt{\alpha^2-1}}\right)$$

$$\therefore \quad -\int_0^{\pi} \frac{1}{(\alpha - \cos x)^2}\,dx = -\frac{\pi}{2}(\alpha^2-1)^{-3/2}\cdot(2\alpha)$$

$$\therefore \quad \int_0^{\pi} \frac{dx}{(\alpha - \cos x)^2} dx = \frac{\pi \alpha}{(\alpha^2 - 1)^{3/2}}$$

Putting $\alpha = 2$;

$$\int_0^{\pi} \frac{dx}{(2 - \cos x)^2} = \frac{\pi(2)}{(3)^{3/2}} = \frac{2\pi}{3\sqrt{3}}$$

Q3) Let $I(a) = \int_0^{\pi} \frac{\log(1 + a\cos x)}{\cos x} \cdot dx$ ...(1)

Applying D.U.I.S. w.r.t. a,

$$I'(a) = \int_0^{\pi} \frac{1}{\cos x}\left[\frac{\partial}{\partial a}\log(1 + a\cos x)\right] dx$$

$$= \int_0^{\pi} \frac{1}{\cos x} \cdot \frac{1}{(1 + a\cos x)} \cos x \cdot dx = \int_0^{\pi} \frac{dx}{1 + a\cos x}$$

Let $t = \tan\frac{x}{2}$

$$\therefore \quad dx = \frac{2\,dt}{1+t^2}$$

and $\cos x = \frac{1-t^2}{1+t^2}$

x	0	π
t	0	∞

$$\therefore \quad I'(a) = \int_0^\infty \frac{1}{1+a\left(\frac{1-t^2}{1+t^2}\right)}\frac{2\,dt}{1+t^2} = 2\int_0^\infty \frac{dt}{1+t^2+a-at^2}$$

$$= 2\int_0^\infty \frac{dt}{(1-a)t^2+(1+a)} = \frac{2}{1-a}\int_0^\infty \frac{dt}{t^2+\left(\frac{1+a}{1-a}\right)}$$

$$= \frac{2}{1-a}\left[\sqrt{\frac{1-a}{1+a}}\tan^{-1} t\sqrt{\frac{1-a}{1+a}}\right]_0^\infty$$

$$= \frac{2}{(1-a)}\sqrt{\frac{1-a}{1+a}}\cdot\left(\frac{\pi}{2}-0\right) = \frac{2}{\sqrt{1-a^2}}\ \frac{\pi}{2}$$

$$\therefore \quad I'(a) = \frac{\pi}{\sqrt{1-a^2}}$$

Integrating w.r.t. a,

$$I(a) = \pi\int\frac{da}{\sqrt{1-a^2}} + A$$

$$\therefore \quad I(a) = \pi\sin^{-1} a + A$$

To find A, we put a = 0 in (1), (2)

$$\therefore \quad I(0) = \int_0^\pi \frac{\log(1)}{\cos x}\,dx = 0$$

and $I(0) = 0 + A$

$$\therefore \quad A = 0$$

$\therefore$ From (2),

$$I(a) = \pi\sin^{-1} a$$

Sol. :

Q4) Let $I(a) = \int_0^{\pi/2} \frac{dx}{1 + a\cos^2 x}, \quad |a| < 1$ ...(1)

Consider,

$$\frac{1}{1 + a\cos^2 x} = \frac{1}{1 + a\left(\frac{1 + \cos 2x}{2}\right)} = \frac{2}{(a+2) + a\cos 2x}$$

Let $b = a + 2$,

$$= \frac{2}{b + a\cos 2x}$$

$\therefore$ From (1),

$$I(a) = \int_0^{\pi/2} \frac{2\,dx}{b + a\cos 2x}$$

Let $2x = t$

$$\therefore \quad 2\,dx = dt$$

x	0	$\pi/2$
t	0	π

$$\therefore \quad I(a) = 2\int_0^{\pi} \frac{1}{b + a\cos t}\frac{dt}{2} = \int_0^{\pi} \frac{dt}{b + a\cos t}$$

Let $\tan\frac{t}{2} = u$

$$\therefore \quad dt = \frac{2\,du}{1 + u^2}$$

and

$$I(a) = \int_0^{\infty} \frac{1}{b + a\left(\frac{1 - u^2}{1 + u^2}\right)} \cdot \frac{2\,du}{(1 + u^2)}$$

$$= 2\int_0^{\infty} \frac{du}{(b + a) + (b - a)u^2}$$

$$\therefore \quad b = a + 2$$

$$= 2\int_0^\infty \frac{du}{(2a+2)+2u^2} = \int_0^\infty \frac{du}{u^2+(a+1)}$$

$$= \left[\frac{1}{\sqrt{a+1}}\tan^{-1}\frac{u}{\sqrt{a+1}}\right]_0^\infty = \frac{1}{\sqrt{a+1}}\left(\frac{\pi}{2}-0\right)$$

$$\therefore \quad I(a) = \frac{\pi}{2\sqrt{a+1}}.$$

i.e. $$\int_0^{\pi/2} \frac{dx}{1+a\cos^2 x} = \frac{\pi}{2\sqrt{a+1}}.$$

Differentiate both sides w.r.t. a and applying D.U.I.S. on left,

$$\int_0^{\pi/2} \frac{\partial}{\partial a}\left[\frac{1}{(1+a\cos^2 x)}\right] dx = \frac{\pi}{2}\frac{d}{da}\left[(a+1)^{-1/2}\right]$$

$$\therefore \int_0^{\pi/2} \frac{-\cos^2 x}{(1+a\cos^2 x)^2}\, dx = \frac{\pi}{2}\left[\left(-\frac{1}{2}\right)(a+1)^{-3/2}\right]$$

Putting $a = \frac{1}{3}$; we get,

$$\int_0^{\pi/2} \frac{\cos^2 x}{\left(1+\frac{1}{3}\cos^2 x\right)^2}\, dx = \frac{\pi}{4}\left(\frac{1}{3}+1\right)^{-3/2}$$

$$\therefore \int_0^{\pi/2} \frac{9\cos^2 x}{(3+\cos^2 x)^2}\, dx = \frac{\pi}{4}\cdot\frac{3}{4}\frac{\sqrt{3}}{2}$$

$$\therefore \int_0^{\pi/2} \frac{\cos^2 x}{(3+\cos^2 x)^2}\, dx = \frac{\pi\sqrt{3}}{96}$$

Sol. :

Q5)

Let $$I(m) = \int_0^\infty \frac{1 - \cos mx}{x} \cdot e^{-x} \cdot dx \qquad \ldots(1)$$

Applying D.U.I.S. w.r.t. m, we get,

$$I'(m) = \int_0^\infty \frac{e^{-x}}{x} \frac{\partial}{\partial m} [1 - \cos mx] \cdot dx$$

$$= \int_0^\infty \frac{e^{-x}}{x} \cdot x \cdot \sin mx \cdot dx = \int_0^\infty e^{-x} \cdot \sin mx \cdot dx$$

$$= \frac{1}{(1+m^2)} \left[-e^{-x} \sin mx - m\, e^{-x} \cos mx \right]_0^\infty = \frac{1}{(1+m^2)} \cdot m$$

Integrating w.r.t. m;

$$I(m) = \int \frac{m}{m^2+1}\, dm + A$$

$$\therefore \quad I(m) = \frac{1}{2} \log(m^2+1) + A \qquad \ldots(2)$$

To find A; we put m = 0 in (1) and (2);

$$I(0) = \int_0^\infty \frac{1-1}{x} e^{-x} \cdot dx = 0$$

and $$I(0) = \frac{1}{2} \log(1) + A = A$$

$$\therefore \quad A = 0$$

$\therefore$ From (2),

$$I(m) = \frac{1}{2} \log(m^2+1)$$

Q6)

Let $$I(b) = \int_0^{\pi/2} \log\left(\frac{a + b \sin\theta}{a - b \sin\theta}\right) \cdot \operatorname{cosec}\theta \cdot d\theta$$

$$= \int_0^{\pi/2} [\log(a + b\sin\theta) - \log(a - b\sin\theta)] \cdot \operatorname{cosec}\theta \cdot d\theta$$

Applying D.U.I.S. w.r.t. b, we get,

$$I'(b) = \int_0^{\pi/2} \left[\frac{\sin\theta}{a + b\sin\theta} - \frac{(-\sin\theta)}{a - b\sin\theta}\right] \operatorname{cosec}\theta \cdot d\theta$$

$$= \int_0^{\pi/2} \left[\frac{1}{a + b\sin\theta} + \frac{1}{a - b\sin\theta}\right] d\theta$$

Let $\tan\frac{\theta}{2} = t$

$$\therefore \quad d\theta = \frac{2\,dt}{(1 + t^2)}$$

and $\sin\theta = \dfrac{2t}{(1 + t^2)}$

θ	0	π / 2
t	0	1

$$\therefore \quad I'(b) = \int_0^1 \left[\frac{1}{a + b\left(\frac{2t}{1 + t^2}\right)} + \frac{1}{a - b\left(\frac{2t}{1 + t^2}\right)}\right] \frac{2\,dt}{(1 + t^2)}$$

$$= 2\int_0^1 \left[\frac{1}{at^2 + 2bt + a} + \frac{1}{at^2 - 2bt + a}\right] dt$$

$$= \frac{2}{a}\int_0^1 \left[\frac{1}{t^2 + \frac{2b}{a} + \frac{b^2}{a^2} - \frac{b^2}{a^2} + 1} + \frac{1}{t^2 - \frac{2b}{a}t + \frac{b^2}{a^2} - \frac{b^2}{a^2} + 1}\right] dt$$

$$= \frac{2}{a}\int_0^1 \left[\frac{1}{\left(t + \frac{b}{a}\right)^2 + \frac{a^2 - b^2}{a^2}} + \frac{1}{\left(t - \frac{b}{a}\right)^2 + \frac{a^2 - b^2}{a^2}}\right] dt$$

Let $k^2 = \dfrac{a^2 - b^2}{a^2}$; so that,

$$I'(b) = \frac{2}{a}\int_0^1 \left[\frac{1}{\left(t+\frac{b}{a}\right)^2 + k^2} + \frac{1}{\left(t-\frac{b}{a}\right)^2 + k^2}\right] dt$$

$$= \frac{2}{a}\left[\frac{1}{k}\tan^{-1}\left(\frac{t+\frac{b}{a}}{k}\right) + \frac{1}{k}\tan^{-1}\left(\frac{t-\frac{b}{a}}{k}\right)\right]_0^1$$

$$= \frac{2}{ak}\left[\tan^{-1}\left(\frac{at+b}{ak}\right) + \tan^{-1}\left(\frac{at-b}{ak}\right)\right]_0^1$$

$$= \frac{2}{ak}\left[\tan^{-1}\left(\frac{a+b}{ak}\right) + \tan^{-1}\left(\frac{a-b}{ak}\right) - \tan^{-1}\left(\frac{b}{ak}\right) - \tan^{-1}\left(\frac{-b}{ak}\right)\right]$$

$$= \frac{2}{ak}\tan^{-1}\left[\frac{\frac{a+b}{ak} + \frac{a-b}{ak}}{1 - \left(\frac{a+b}{ak}\right)\left(\frac{a-b}{ak}\right)}\right] = \frac{2}{ak}\tan^{-1}\left[\frac{\frac{2a}{ak}}{1 - \left(\frac{a^2-b^2}{a^2k^2}\right)}\right]$$

$$= \frac{2}{ak}\tan^{-1}\left[\frac{2/k}{1 - \left(\frac{a^2-b^2}{a^2-b^2}\right)}\right]$$

$$(\because a^2k^2 = a^2 - b^2)$$

$$= \frac{2}{ak}\cdot\frac{\pi}{2}$$

$$= \frac{\pi}{ak} = \frac{\pi}{\sqrt{a^2-b^2}} \qquad \left(\because k = \frac{\sqrt{a^2-b^2}}{a}\right)$$

$$\therefore \quad I'(b) = \frac{\pi}{\sqrt{a^2-b^2}}$$

Integrating w.r.t. b,

$$I(b) = \pi\int \frac{db}{\sqrt{a^2-b^2}} + A$$

$$I(b) = \pi \sin^{-1}\frac{b}{a} + A \qquad \text{...(2)}$$

To find A; we put b = 0 in (1) and (2),

$$I(0) = \int_0^{\pi/2} \log\left(\frac{a}{a}\right) \operatorname{cosec}\theta \cdot d\theta = 0$$

and $I(0) = 0 + A = A$

$\therefore \quad A = 0$

From (2),

$$I(b) = \pi \sin^{-1}\left(\frac{b}{a}\right)$$

Q7)

Sol. : We regard a as a parameter.

Let $I(a) = \int_0^\infty \frac{e^{-ax} \cdot \sin px}{x} \cdot dx$ ...(1)

Applying D.U.I.S. w.r.t. a,

$$I'(a) = \int_0^\infty \frac{\sin px}{x} \frac{\partial}{\partial a}\left(e^{-ax}\right) \cdot dx$$

$$= \int_0^\infty \frac{\sin px}{x}\left(-x e^{-ax}\right) \cdot dx = -\int_0^\infty e^{-ax} \cdot \sin px \cdot dx$$

$$= -\frac{1}{(a^2 + p^2)}\left[-a e^{-ax} \sin px - p e^{-ax} \cos px\right]_0^\infty$$

$$= -\frac{1}{(a^2+p^2)}[p] = -\frac{p}{a^2+p^2}$$

integrating w.r.t. a,

$$I(a) = -p\int \frac{da}{a^2+p^2} + A = -p \cdot \frac{1}{p}\tan^{-1}\left(\frac{a}{p}\right) + A$$

$$\therefore \quad I(a) = -\tan^{-1}\left(\frac{a}{p}\right) + A \qquad \ldots(2)$$

putting $a = 0$ in (1), (2); we get,

$$I(0) = \int_0^\infty \frac{\sin px}{x} \cdot dx = \frac{\pi}{2} \text{ (standard result)}$$

$$\text{and} \quad I(0) = 0 + A$$

$$\therefore \quad A = \frac{\pi}{2}$$

$$\therefore \text{ From (2),} \quad I(a) = \frac{\pi}{2} - \tan^{-1}\left(\frac{a}{p}\right) = \tan^{-1}\left(\frac{p}{a}\right)$$

$$\left(\because \cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x \text{ and } \cot^{-1} x = \tan^{-1}\frac{1}{x}\right)$$

Ex. 8 : Prove that :

$$\int_0^a \frac{\log(1+ax)}{1+x^2}dx = 1/2 \tan^{-1} a\,[\log(1+a^2)] \text{ and hence deduce that :}$$

$$\int_0^1 \frac{\log(1+x)}{1+x^2}dx = \frac{\pi}{8}\log 2.$$

(Dec. 99, 6 Marks)

Sol. :

$$\text{Let} \quad I(a) = \int_0^a \frac{\log(1+ax)}{1+x^2} \cdot dx$$

Applying Leibnitz' formula, we get

$$I'(a) = \int_0^a \frac{\partial}{\partial a}\frac{\log(1+ax)}{1+x^2} \cdot dx + \frac{\log(1+a^2)}{1+a^2}\frac{da}{da} - \frac{\log(1+0)}{1+0}\frac{d}{da}(0)$$

$$= \int_0^a \frac{1}{(1+x^2)}\frac{1}{(1+ax)} \cdot x \cdot dx + \frac{\log(1+a^2)}{(1+a^2)} \qquad \ldots(1)$$

We evaluate the integral using partial fractions. Consider

$$\frac{x}{(1+ax)(1+x^2)} = \frac{A}{1+ax} + \frac{Bx+c}{1+x^2} \quad \ldots (2)$$

$$\therefore \quad x \equiv A(1+x^2) + (Bx+C)(1+ax)$$

Putting $x = -\frac{1}{a}$, we get

$$-\frac{1}{a} = A\left(1+\frac{1}{a^2}\right)$$

$$\therefore \quad A = \frac{-a}{a^2+1}$$

Comparing coefficients of x^2 on both sides,

$$0 = A + aB$$

$$\therefore \quad B = \frac{1}{a^2+1}$$

Comparing coefficients of x on both sides,

$$1 = B + aC$$

$$\therefore \quad C = \frac{a}{a^2+1}$$

Hence, (2) becomes,

$$= \frac{x}{(1+ax)(1+x^2)} = \frac{-a}{(a^2+1)(1+ax)} + \frac{x+a}{(a^2+1)(x^2+1)}$$

$$\therefore \quad \int_0^a \frac{x}{(1+ax)(1+x^2)} \cdot dx = \left[\frac{-a}{(a^2+1)(1+ax)} + \frac{x+a}{(a^2+1)(x^2+1)}\right] dx$$

$$= -\frac{a}{a^2+1}\int_0^a \frac{dx}{1+ax} + \frac{1}{a^2+1}\int_0^a \frac{x}{x^2+1}\,dx + \frac{a}{a^2+1}\int_0^a \frac{dx}{x^2+1}$$

$$= -\frac{a}{a^2+1}\left[\frac{1}{a}\log(1+ax)\right]_0^a + \frac{1}{a^2+1}\left[\frac{1}{2}\log(x^2+1)\right]_0^a + \frac{a}{a^2+1}\left[\tan^{-1}x\right]_0^a$$

$$= -\frac{1}{(a^2+1)}\left[\log(1+a^2) - \log(1)\right] + \frac{1}{2(a^2+1)}[\log(1+a^2)] + \frac{a}{a^2+1}\tan^{-1}a$$

$$= -\frac{1}{(a^2+1)}\log(1+a^2) + \frac{1}{2(a^2+1)}\log(1+a^2) + \frac{a}{a^2+1}\tan^{-1}a$$

$$= -\frac{1}{2(a^2+1)}\log(1+a^2) + \frac{a}{a^2+1}\tan^{-1}a$$

Hence, from (1)

$$I'(a) = -\frac{1}{2(a^2+1)} + \frac{a}{a^2+1}\tan^{-1} a + \frac{1}{a^2+1}\log(1+a^2)\log(1+a^2)$$

$$= \frac{1}{(a^2+1)}\left[\frac{1}{2}\log(1+a^2) + a\tan^{-1} a\right]$$

Integrating w.r.t. a, we get

$$I(a) = \frac{1}{2}\int \frac{\log(1+a^2)}{1+a^2}da + \int \frac{a}{a^2+1}\tan^{-1}a + A \quad \text{... (3)}$$

Consider, $\int \frac{a}{a^2+1}\tan^{-1}a\cdot da$

Integrating by parts, we get

$$= \tan^{-1}a\left[\frac{1}{2}\log(a^2+1)\right] - \int \frac{1}{1+a^2}\left[\frac{1}{2}\log(a^2+1)\right]da$$

$$= \frac{1}{2}\tan^{-1}a.\log(a^2+1) - \frac{1}{2}\int \frac{\log(1+a^2)}{1+a^2}da$$

Hence, from (3),

$$I(a) = \frac{1}{2}\int \frac{\log(1+a^2)}{1+a^2}da + \frac{1}{2}\tan^{-1}a.\log(a^2+1) - \frac{1}{2}\int \frac{\log(1+a^2)}{1+a^2}da + A$$

$$\therefore I(a) = \frac{1}{2}\tan^{-1}a\cdot\log(a^2+1) + A \quad \text{... (4)}$$

Putting $a = 0$ in (4) and in the given integral, we get

$I(0) = 0$ and

$I(0) = 0 + A = A$

$\therefore \quad A = 0$

$\because$ From (4), $I(a) = \int_0^a \frac{\log(1+ax)}{1+x^2}\cdot dx = \frac{1}{2}\tan^{-1} a\cdot\log(a^2+1)$

Putting $a = 1$, we get

$$\int_0^a \frac{\log(1+x)}{1+x^2}dx = \frac{1}{2}\tan^{-1}(1)\log(2) = \frac{\pi}{8}\log 2.$$

Q9) Let, $$I(a) = \int_0^{\pi/2} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} \quad ...(1)$$

Applying D.U.I.S. w.r.t. a,

$$I'(a) = \int_0^{\pi/2} \left(\frac{\partial}{\partial a} \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} \right) \cdot dx$$

$$= \int_0^{\pi/2} \frac{-2a \sin^2 x}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} \cdot dx \quad ...(2)$$

$$= -\frac{2a}{b^4} \int_0^{\pi/2} \frac{\sin^2 x}{\cos^4 x \left(\frac{a^2}{b^2} \tan^2 x + 1 \right)^2} \cdot dx$$

$$= \frac{-2a}{b^4} \int_0^{\pi/2} \frac{\tan x \cdot (\tan x \cdot \sec^2 x \cdot dx)}{\left(1 + \frac{a^2}{b^2} \tan^2 x \right)^2}$$

Let $\frac{a^2}{b^2} \tan^2 x = u$

$\therefore \quad \frac{a^2}{b^2} 2 \tan x \cdot \sec^2 x \cdot dx = du$

$\therefore \quad \tan x \cdot \sec^2 x \cdot dx = \frac{b^2}{2a^2} du$

And, $\tan x = \frac{b}{a} \sqrt{u}$

x	0	$\pi/2$
u	0	∞

$$\therefore \quad I'(a) = -\frac{2a}{b^4} \int_0^{\infty} \frac{\left(\frac{b}{a} \sqrt{u} \right) \left(\frac{b^2}{2a^2} \right) du}{(1+u)^2}$$

$$= -\frac{2a}{b^4} \cdot \frac{b}{a} \cdot \frac{b^2}{2a^2} \int_0^{\infty} \frac{u^{3/2-1}}{(1+u)^{3/2+1/2}} \cdot du$$

$$= -\frac{1}{a^2 b}\,\beta\left(\frac{3}{2},\frac{1}{2}\right) = -\frac{1}{a^2 b}\frac{\Gamma(3/2)\,\Gamma(1/2)}{\Gamma(2)} = -\frac{1}{a^2 b}\cdot\frac{\frac{1}{2}\sqrt{\pi}\cdot\sqrt{\pi}}{1}$$

$$\therefore \quad I'(a) = -\frac{\pi}{2a^2 b} \qquad ...(3)$$

Integrating w.r.t. a,

$$I(a) = -\frac{\pi}{2b}\int \frac{1}{a^2}\,da + A$$

$$I(a) = \frac{\pi}{2ab} + A \qquad ...(4)$$

Putting a = b in (1) and (4); we get,

$$I(b) = \int_0^{\pi/2} \frac{dx}{b^2} = \frac{\pi}{2b^2} \quad \text{and} \quad I(b) = \frac{\pi}{2b^2} + A$$

$$\therefore \quad \frac{\pi}{2b^2} = \frac{\pi}{2b^2} + A \qquad \therefore \quad A = 0$$

$\therefore$ From (4),

$$I(a) = \frac{\pi}{2ab} \qquad ...(5)$$

From (2) and (3); we write,

$$\int_0^{\pi/2} \frac{-2a\sin^2 x}{(a^2\sin^2 x + b^2\cos^2 x)^2}\,dx = -\frac{\pi}{2a^2 b}$$

$$\therefore \quad \int_0^{\pi/2} \frac{\sin^2 x}{(a^2\sin^2 x + b^2\cos^2 x)^2}\,dx = \frac{\pi}{4a^3 b} \qquad ...(6)$$

Similarly, we can show that,

$$\int_0^{\pi/2} \frac{\cos^2 x}{(a^2\sin^2 x + b^2\cos^2 x)^2}\,dx = \frac{\pi}{4ab^3} \qquad ...(7)$$

Adding (6) and (7), we have

$$\int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{(a^2\sin^2 x + b^2\cos^2 x)^2}\,dx = \frac{\pi}{4a^3 b} + \frac{\pi}{4ab^3}$$

i.e.,

$$\int_0^{\pi/2} \frac{dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} = \frac{\pi}{4ab}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$$

sign.

Q10)

$$\frac{dI}{d\alpha} = \int_0^\infty \frac{\partial f}{\partial \alpha} dx = \int_0^{\pi/2} \frac{-\sin\alpha\cos x}{(1+\cos\alpha\cos x)\cos x} dx$$

$$= \int_0^{\pi/2} \frac{-\sin\alpha}{1+\cos\alpha\cos x} dx$$

Put $t = \tan\frac{x}{2}$ $\therefore$ $dx = \frac{2\,dt}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

When $x = 0$, $t = 0$; when $x = \pi/2$, $t = 1$.

$$\therefore \frac{dI}{d\alpha} = \int_0^1 \frac{-\sin\alpha}{1+\cos\alpha\cdot\frac{(1-t^2)}{(1+t^2)}}\cdot\frac{2\,dt}{1+t^2}$$

$$= -\sin\alpha \int_0^1 \frac{2\,dt}{(1+t^2)+\cos\alpha(1-t^2)}$$

$$= -\sin\alpha \int_0^1 \frac{2\,dt}{(1+\cos\alpha)+(1-\cos\alpha)t^2}$$

$$= \frac{-2\sin\alpha}{(1-\cos x)} \int_0^1 \frac{dt}{[(1+\cos\alpha)/(1-\cos\alpha)]+t^2}$$

$$= \frac{-2\sin\alpha}{(1-\cos x)}\cdot\frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}}\left[\tan^{-1}\left\{\frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}}\right\}\cdot t\right]_0^1$$

$$= \frac{-2\sin\alpha}{\sqrt{1-\cos\alpha}\,\sqrt{1+\cos\alpha}}\left[\tan^{-1}\left\{\frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}}\right\}\right]$$

$$= \frac{-2\sin\alpha}{\sqrt{1-\cos^2\alpha}}\left[\tan^{-1}\sqrt{\frac{2\sin^2(\alpha/2)}{2\cos^2(\alpha/2)}}\right]$$

$$= -2\tan^{-1}\tan\frac{\alpha}{2} = -2\frac{\alpha}{2} = -\alpha$$

$$\therefore \quad \frac{dI}{d\alpha} = -\alpha$$

Integrating both sides w.r.t. α,

$$I = -\frac{\alpha^2}{2} + c \qquad \text{..........(1)}$$

To find c, we put $\alpha = \pi/2$. **(Note this)**

$$\therefore \quad I\left(\frac{\pi}{2}\right) = -\frac{\pi^2}{8} + c$$

Now, putting $\alpha = \pi/2$ in the given integral,

$$I\left(\frac{\pi}{2}\right) = \int_0^{\pi/2} 0 \cdot dx = 0 \qquad \therefore \quad c = \frac{\pi^2}{8}$$

Hence, from (1),

$$I = -\frac{\alpha^2}{2} + \frac{\pi^2}{8} = \frac{\pi^2}{8} - \frac{\alpha^2}{2}.$$

$$I(a) = \int_0^1 x^a\,dx = \left[\frac{x^{a+1}}{a+1}\right]_0^1 = \frac{1}{a+1}$$

Differentiating w.r.t. a, we get,

$$\frac{dI}{da} = \int_0^1 x^a \log x \cdot dx = -\frac{1}{(a+1)^2}$$

$$\frac{d^2 I}{da^2} = \int_0^a x^a (\log x)^2 \, dx = \frac{(-1)^2 \cdot 2}{(a+1)^2}$$

Thus, differentiating n times

$$\frac{d^n I}{da^n} = \int_0^a x^a (\log x)^n \, dx = \frac{(-1)^n \cdot n!}{(a+1)^n}.$$

Ex. 19: Show that $\int_0^{\pi} \log(1 - a\cos x)\, dx = \pi \log\left[\frac{1+\sqrt{1-a^2}}{2}\right], |a| < 1.$

(M.U. 2002)

Sol. : Let $I(a) = \int_0^{\pi} \log(1 - a\cos x)\, dx$

$$\therefore \frac{dI(a)}{da} = \int_0^{\pi} \frac{\partial f}{\partial a} dx = \int_0^{\pi} \frac{. - \cos x}{1 - a\cos x} dx$$

$$= \frac{1}{a}\int_0^{\pi} \frac{-a\cos x}{1 - a\cos x} dx = \frac{1}{a}\int_0^{\pi} \frac{(1 - a\cos x) - 1}{1 - a\cos x} dx$$

$$= \frac{1}{a}\int_0^{\pi} \left(1 - \frac{1}{1 - a\cos x}\right) dx = \frac{1}{a}\left[\pi - \int_0^{\pi} \frac{dx}{1 - a\cos x}\right]$$

To evaluate the integral, put $t = \tan\left(\frac{x}{2}\right)$

$$\therefore \int_0^{\pi} \frac{dx}{1 - a\cos x} = \int_0^{\infty} \frac{1}{1 - a \cdot \left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2\,dt}{1+t^2}$$

$$= \int_0^{\infty} \frac{2\,dt}{(1-a) + (1+a)t^2} = \frac{2}{(1+a)} \int_0^{\infty} \frac{dt}{\left(\frac{1-a}{1+a}\right) + t^2} dt$$

$$= \frac{2}{(1+a)} \sqrt{\frac{1+a}{1-a}} \left[\tan^{-1}\left(t \cdot \sqrt{\frac{1+a}{1-a}}\right)\right]_0^{\infty} = \frac{\pi}{\sqrt{1-a^2}}$$

$$\therefore \frac{dI(a)}{da} = \frac{1}{a}\left[\pi - \frac{\pi}{\sqrt{1-a^2}}\right] = \pi\left[\frac{1}{a} - \frac{1}{a\sqrt{1-a^2}}\right]$$

Integrating w.r.t. a, we get,

$$I(a) = \pi\left[\log a - \int \frac{da}{a\sqrt{1-a^2}}\right] + c$$

To find the integral put $a = \sin\theta$.

$$\therefore \int \frac{da}{a\sqrt{1-a^2}} = \int \frac{\cos\theta}{\sin\theta \cdot \cos\theta}\, d\theta = \int \operatorname{cosec}\theta\, d\theta$$

$$= \log(\operatorname{cosec}\theta - \cot\theta) = \log\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \log\left(\frac{1-\sqrt{1-a^2}}{a}\right) \quad [\because \ \sin\theta = a]$$

$$\therefore \ I(a) = \pi\left[\log a - \log\left(\frac{1-\sqrt{1-a^2}}{a}\right)\right] + c$$

$$= \pi \cdot \log\left(\frac{a^2}{1-\sqrt{1-a^2}}\right) + c$$

$$= \pi \cdot \log\left(\frac{a^2}{1-\sqrt{1-a^2}} \cdot \frac{1+\sqrt{1-a^2}}{1+\sqrt{1-a^2}}\right) + c$$

$$= \pi \cdot \log(1+\sqrt{1-a^2}) + c$$

To find c put $a = 0$ $\quad \therefore \ I(0) = \pi \log 2 + c$

But $I(0) = \int_0^{\pi} \log 1\, dx = 0$ [By data]

$$\therefore \ c = -\pi \log 2$$

$$\therefore \ I(a) = \pi \cdot \left[\log(1+\sqrt{1-a^2}) - \log 2\right]$$

$$= \pi \cdot \log\left(\frac{1+\sqrt{1-a^2}}{2}\right).$$

Q14)

Sol. : We have $\int_0^{\infty} e^{-xy}\, dx = \left[\dfrac{e^{-xy}}{-y}\right]_0^{\infty}$

$$= \left[\frac{0-1}{-y}\right] = \frac{1}{y}$$

Let $I(a) = \int_0^{\infty} \dfrac{e^{-ax} - e^{-bx}}{x}\, dx$

$$\therefore \quad \frac{dI}{da} = \int_0^{\infty} \frac{\partial f}{\partial a}\, dx = \int_0^{\infty} -\frac{e^{-ax} \cdot x}{x}\, dx$$

$$= -\int_0^\infty e^{-ax}\,dx = -\frac{1}{a} \qquad \text{[By (1)]}$$

$$\therefore \quad dI = -\frac{da}{a}$$

Integrating both sides we get,

$$I(a) = -\log a + c$$

To find c, we put $a = b$

$$I(b) = \int_0^\infty \frac{e^{-ax} - e^{-ax}}{x}\,dx = 0$$

$$\therefore \quad c = \log b$$

$$\therefore \quad I = \log b - \log a = \log\left(\frac{b}{a}\right).$$

Q15) **Sol.** : Let $I(a) = \int_0^\infty \frac{\tan^{-1}(x/a) - \tan^{-1}(x/b)}{x} dx$

$\therefore$ By the rule of differentiation under the integral sign,

$$\frac{dI}{da} = \int_0^\infty \frac{\partial f}{\partial a} dx = \int_0^\infty \frac{1}{1+(x^2/a^2)}\left(-\frac{x}{a^2}\right)\cdot\frac{1}{x} dx$$

$$= -\int_0^\infty \frac{dx}{a^2+x^2} = -\left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^\infty = -\frac{\pi}{2a}$$

Integrating both sides, we get $I = -\frac{\pi}{2}\log a + c$

To find c, we put $a = b$. $\therefore$ $I(b) = -\frac{\pi}{2}\log b + c$

But $I(a) = \int_0^\infty \frac{\tan^{-1}(x/a) - \tan^{-1}(x/b)}{x} dx$

$\therefore$ $I(b) = \int_0^\infty 0\, dx = 0$

$$0 = -\frac{\pi}{2}\log b + c \quad \therefore\ c = \frac{\pi}{2}\log b$$

$$\therefore\ I = \frac{\pi}{2}\log b - \frac{\pi}{2}\log a = \frac{\pi}{2}\log\frac{b}{a}.$$

Q16)

Sol.: Let $I = \int_0^{\pi/2} \log(a^2 \cos^2\theta + b^2 \sin^2\theta)\, d\theta$

$$\therefore \quad \frac{dI}{da} = \int_0^{\pi/2} \frac{\partial f}{\partial a}\, d\theta = \int_0^{\pi/2} \frac{2a\cos^2\theta\, d\theta}{a^2\cos^2\theta + b^2\sin^2\theta}$$

$$= \int_0^{\pi/2} \frac{2a}{a^2 + b^2\tan^2\theta}\, d\theta$$

Put $t = \tan\theta$

$$\therefore \quad dt = \sec^2\theta\, d\theta = (1 + \tan^2\theta)\, d\theta = (1 + t^2)\, d\theta$$

$$\therefore \quad \frac{dI}{da} = \int_0^\infty \frac{2a}{(a^2 + b^2 t^2)} \cdot \frac{dt}{(1+t^2)}$$

$$= \frac{2a}{a^2 - b^2} \int_0^\infty \left[\frac{1}{1+t^2} - \frac{b^2}{a^2 + b^2 t^2} \right] dt \qquad \text{[By partial fractions]}$$

$$= \frac{2a}{a^2 - b^2} \left[\tan^{-1} t - \frac{b}{a} \tan^{-1}\left(t \cdot \frac{b}{a} \right) \right]_0^\infty$$

$$= \frac{2a}{a^2 - b^2} \left[\frac{\pi}{2} - \frac{b}{a} \cdot \frac{\pi}{2} \right] = \frac{\pi}{a+b}$$

Integrating w.r.t. a, we get,

$$I = \pi \log(a+b) + c \qquad \text{.......... (i)}$$

To find c, we put $a = b$,

$$\therefore \quad I(b) = \int_0^{\pi/2} \log b^2 \cdot d\theta = \log b^2 \cdot [\theta]_0^{\pi/2}$$

$$= \pi \cdot \log b$$

And from (i), $\quad I(b) = \pi \log 2b + c$

$$\therefore \quad \pi \log b = \pi \log 2b + c$$

$$\therefore \quad c = \pi \log\left(\frac{b}{2b}\right) = \pi \log\left(\frac{1}{2}\right)$$

$$\therefore \quad I = \pi \log(a+b) + \pi \log\left(\frac{1}{2}\right)$$

$$= \pi \log\left(\frac{a+b}{2}\right).$$

Ex.: Evaluate $\int_0^\pi \frac{dx}{a + b\cos x}$, $\quad a > 0,\ b > 0$

and deduce that $\int_0^\pi \frac{dx}{(a + b\cos x)^2} = \frac{\pi a}{(a^2 - b^2)^{3/2}}$

and $\int_0^\pi \frac{\cos x\, dx}{(a + b\cos x)^2} = -\frac{\pi b}{(a^2 - b^2)^{3/2}}$. **(M.U. 1995, 98, 2003)**

Sol.: Let $I = \int_0^\pi \frac{dx}{a + b\cos x}$

Putting $t = \tan\frac{x}{2}$, $\quad dt = \frac{1}{2}\sec^2\frac{x}{2}\, dx$

$$\therefore \quad dt = \frac{1}{2}(1+t^2)\, dx \qquad \therefore \quad dx = \frac{2\,dt}{1+t^2}$$

When $x = 0$, $t = 0$; when $x = \pi$, $t = \infty$.

$$\therefore \quad I = \int_0^{\infty} \frac{2\,dt/(1+t^2)}{a + b \cdot \dfrac{1-t^2}{1+t^2}} = 2\int_0^{\infty} \frac{dt}{(a+b)+(a-b)t^2}$$

$$= \frac{2}{a-b}\int_0^{\infty} \frac{dt}{t^2 + \left(\dfrac{a+b}{a-b}\right)}$$

$$= \frac{2}{(a-b)}\sqrt{\left(\frac{a-b}{a+b}\right)}\left[\tan^{-1}\left(t \cdot \sqrt{\left(\frac{a-b}{a+b}\right)}\right)\right]_0^{\infty}$$

$$= \frac{2}{\sqrt{a^2-b^2}} \cdot \frac{\pi}{2} = \frac{2}{\sqrt{a^2-b^2}}$$

$$\therefore \quad \int_0^{\pi} \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2-b^2}} \qquad \text{.......... (i)}$$

Now we apply the rule of differentiation under the integral sign. Differentiating both sides w.r.t a, we get,

$$\int_0^{\pi}\left[\frac{\partial}{\partial a}\left(\frac{1}{a+b\cos x}\right)\right]dx = -\frac{\pi}{2}\cdot(a^2-b^2)^{-3/2}\cdot 2a$$

$$\therefore \quad \int_0^{\pi} -\frac{1}{(a+b\cos x)^2}\,dx = -\frac{\pi a}{(a^2-b^2)^{3/2}}$$

$$\therefore \quad \int_0^{\pi} \frac{dx}{(a+b\cos x)^2} = \frac{\pi a}{(a^2-b^2)^{3/2}}$$

Again differentiating (i) w.r.t b,

$$\int_0^{\pi}\left[\frac{\partial}{\partial b}\left(\frac{1}{a+b\cos x}\right)\right]dx = -\frac{\pi}{2}(a^2-b^2)^{3/2}\cdot(-2b)$$

$$\therefore \quad \int_0^{\pi} -\frac{1}{(a+b\cos x)^2}\cdot\cos x\cdot dx = \frac{\pi b}{(a^2-b^2)^{3/2}}$$

$$\therefore \quad \int_0^{\pi} \frac{\cos x}{(a+b\cos x)^2}\,dx = -\frac{\pi b}{(a^2-b^2)^{3/2}}$$

Q18)

Sol. : Let $I(\alpha) = \int_0^{\pi/2} \frac{\log(1+\cos\alpha\cos x)}{\cos x}\,dx$

$$\therefore \quad \frac{dI(\alpha)}{d\alpha} = \int_0^{\infty} \frac{\partial f}{\partial \alpha}\,dx = \int_0^{\pi/2} \frac{-\cos x \cdot \sin\alpha}{(1+\cos\alpha\cos x)\cos x} \cdot dx$$

$$= -\int_0^{\pi/2} \frac{\sin\alpha}{1+\cos\alpha\cos x}\,dx$$

Putting $a = 1$ and $b = \cos\alpha$ in

$$\int_0^{\pi/2} \frac{dx}{a+b\cos x} = \frac{1}{\sqrt{a^2-b^2}} \cdot \cos^{-1}\left(\frac{b}{a}\right), \text{ we get}$$

$$\frac{dI(\alpha)}{d\alpha} = -\int_0^{\pi/2} \frac{\sin\alpha}{1+\cos\alpha\cos x}\,dx$$

$$= \frac{-\sin\alpha}{\sqrt{1-\cos^2\alpha}} \cdot \cos^{-1}(\cos\alpha) = -\alpha$$

$$\therefore \quad dI(\alpha) = -\alpha\, d\alpha$$

Integrating both sides we get,

$$I(\alpha) = -\frac{\alpha^2}{2} + c$$

To find c, we put $\alpha = \pi/2$ **[Note this]**

But $I(\alpha) = \int_0^{\pi/2} \frac{\log(1)}{\cos x}\,dx = 0$

$$\therefore \quad 0 = -\frac{\pi^2}{8} + c \qquad \therefore \quad c = \frac{\pi^2}{8}$$

$$\therefore \quad I(\alpha) = \frac{\pi^2}{8} - \frac{\alpha^2}{2}.$$